Bogomolny Yang-Mills-Higgs Solutions in (2+1) anti-de Sitter Space

Theodora Ioannidou

Institute of Mathematics, University of Kent at Canterbury,
Canterbury, CT2 7NF, U.K.
Email : T.Ioannidou@ukc.ac.uk

June 25, 2002

Abstract

This paper investigates an integrable system which is related to hyperbolic monopoles; ie the Bogomolny Yang-Mills-Higgs equations in (2+1) anti-de Sitter space which are integrable and whose solutions can be obtained using analytical methods. In particular, families of soliton solutions have been constructed explicitly and their dynamics has been investigated in some detail.

I. Introduction

Static BPS monopoles are solutions of a nonlinear elliptic partial differential equation on some three-dimensional Riemannian manifold. Most work on monopoles has dealt with the case when this manifold is Euclidean space $\mathbb{R}^3$ since the equations are integrable and geometrical techniques can be applied. [The introduction of time dependence destroys the integrability]. In addition, the monopole equations on hyperbolic space $\mathbb{H}^3$ are also integrable [1] and often hyperbolic monopoles turn out to be easier to study than the Euclidean (see, for example, [2]). Moreover, recently, it has been rigorously established [3] that in the limit as the curvature of hyperbolic space tends to zero then Euclidean monopoles are recovered. In this paper, we consider an integrable system [4] which is related to hyperbolic monopoles and follows from replacing the positive definite space $\mathbb{H}^3$ by a Lorentzian version, ie the anti-de Sitter space. In recent years, the n-dimensional anti-de Sitter spacetime has been of continuing interest since it is the base of M-theory and a source of simple examples studying methods and spacetime concepts both on classical and quantum level. It also arises as the natural ground state of gauged supergravity theories when quantized [5].

The Bogomolny version of Yang-Mills-Higgs equations for Yang-Mills-Higgs fields on a three-dimensional Riemannian manifold ($\mathcal{M}$) with gauge group $SU(2)$ have the form

$$D_i\Phi = \frac{1}{2\sqrt{|g|}} g_{ij}\, \epsilon^{jkl} F_{kl}. \tag{1}$$

Here A_k, for $k = 0, 1, 2$, is the $su(2)$-valued gauge potential, with field strength $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ and $\Phi = \Phi(x^\mu)$ is the $su(2)$-valued Higgs field; while $x^\mu = (x^0, x^1, x^2)$

represent the local coordinates on M. The action of the covariant derivative $D_i = \partial_i + A_i$ on Φ is: $D_i\Phi = \partial_i\Phi + [A_i, \Phi]$. Equation (1) is integrable in the sense that a Lax pair exists for constant curvature. In particular, the solutions of (1) correspond to Euclidean or hyperbolic BPS monopoles when $(\mathcal{M}, g)$ is Euclidean $\mathbb{R}^3$ or hyperbolic $\mathbb{H}^3$ space.

There are two curved spacetimes with constant curvature: (i) the de Sitter space with positive scalar curvature and (ii) the anti-de Sitter space with negative curvature. By definition the (2+1)-dimensional anti-de Sitter space is the universal covering space of the hyperboloid $\mathcal{H}$ satisfied by the equation

$$U^2 + V^2 - X^2 - Y^2 = 1 \tag{2}$$

with metric given by

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2. \tag{3}$$

By parametrizing the hyperboloid $\mathcal{H}$ by

$$\begin{aligned} U &= \sec\rho\cos\theta \\ V &= \sec\rho\sin\theta \\ X &= \tan\rho\cos\phi \\ Y &= \tan\rho\sin\phi \end{aligned} \tag{4}$$

for $\rho \in [0, \pi/2)$, the corresponding metric takes the form

$$ds^2 = \sec^2\rho\left(-d\theta^2 + d\rho^2 + \sin^2\rho\, d\phi^2\right). \tag{5}$$

The spacetime contains closed timelike curves, due to the periodicity of θ (for more details, see Ref. [6]). In fact, anti-de Sitter space (as a manifold) is the product of an open spatial disc with θ and constant curvature equal to minus six; where (ρ, ϕ) correspond to polar coordinates and $\theta \in R$ being the time. Null spacelike infinity $\mathcal{I}$ consists of the timelike cylinder $\rho = \pi/2$ and this surface is never reached by timelike geodesics.

If the Poincaré coordinates (r, x, t) for $r > 0$ are defined as

$$\begin{aligned} r &= \frac{1}{U+X} \\ x &= \frac{Y}{U+X} \\ t &= \frac{-V}{U+X} \end{aligned} \tag{6}$$

the metric simplifies to the following form

$$ds^2 = r^{-2}(-dt^2 + dr^2 + dx^2). \tag{7}$$

Note that, the Poincaré coordinates cover a small part of anti-de Sitter space, ie that corresponding to half of the hyperboloid $\mathcal{H}$ for $U + X > 0$; which is the shaded region in FIG. 1. The surface $r = 0$ is part of infinity $\mathcal{I}$.

Hitchin [7] show that the minitwistor space corresponding to Poincaré space (7) is $CP^1 \times CP^1$ and can be visualized as a quadric Q in CP^3; while the points of spacetime correspond to certain plane sections (conics) of Q with space CP^3. The relevant conics which have to be real and nondegenerate, are given by the expression [4]

$$\omega = v - r^2\,(\mu - u)^{-1} \tag{8}$$

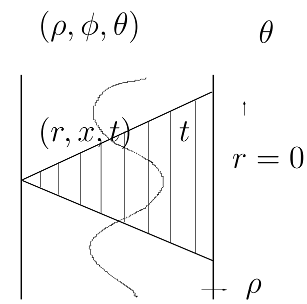

Figure 1: The Penrose diagram of anti-de Sitter space. The boundary of anti-de Sitter is the boundary of the cylinder.

where (ω, μ) are standard coordinates on the two CP^1 factor of Q, while $u = x + t$ and $v = x - t$. Note that the Poincaré coordinates (r, x, t) cover all of the space of these conics (which is the top half of RP^3) expect for a set of measure zero. In order to see the correspondence between spacetime and twistor space Q one needs to substitute (6) into (8).

Consider the set of linear equations

$$\begin{aligned} \left[rD_r - 2(\lambda - u)D_u - \Phi\right]\Psi &= 0 \\ \left[2D_v + \frac{\lambda - u}{r}D_r - \frac{\lambda - u}{r^2}\Phi\right]\Psi &= 0. \end{aligned} \tag{9}$$

Here $\lambda \in \mathbb{C}$ and (r, u, v) are the Poincaré coordinates which cover, only, the shaded region of FIG. 1. The gauge fields (Φ, A_r, A_u, A_v) are 2×2 trace-free matrices depending only on (r, u, v) and $\Psi(\lambda, r, u, t)$ is a unimodular 2×2 matrix function satisfying the reality condition $\Psi(\lambda)\Psi(\bar{\lambda})^\dagger = I$ (where $\dagger$ denotes the complex conjugate transpose). The system (9) is overdetermined and in order for a solution Ψ to exist the following integrability conditions need to be satisfied

$$\begin{aligned} D_u\Phi &= rF_{ur} \\ D_v\Phi &= -rF_{vr} \\ D_r\Phi &= -2rF_{uv}. \end{aligned} \tag{10}$$

The above equations are consistent with the ones obtained from (1) using the Poincaré coordinates.

The gauge and Higgs fields in terms of the function Ψ can be obtained from the Lax pair (9). Note that, as $\lambda \to \infty$ the function Ψ goes to the identity matrix which implies that

$$A_u = 0, \qquad A_r = \frac{1}{r}\,\Phi. \tag{11}$$

On the other hand, for $\lambda = 0$ and using (11) the rest of the gauge fields are defined as

$$\begin{aligned} \Phi &= -\frac{r}{2}\,J_rJ^{-1} - u\,J_uJ^{-1} \\ A_v &= \frac{u}{2r}\,J_rJ^{-1} - J_vJ^{-1} \end{aligned} \tag{12}$$

where $J(r,u,v) \doteq \Psi(\lambda=0,r,u,v)$. Note that, in this case, the first equation of the system (10) is automatically satisfied (due to the specific gauge choice).

Recently, Ward [4] has shown that holomorphic vector bundles V over Q determine multi-soliton solutions of (10) in anti-de Sitter space via the usual Penrose transform. This way a five-parameter family of soliton solutions can be obtained, in a similar way as for flat spacetime [8]. Later, more solutions of equations (10) were obtained by Zhou [9, 10] using Darboux transformations with constant and variable spectral parameters. In what follows, we use the Riemann problem with zeros to construct families of soliton solutions and observe the occurence of different types of scattering behaviour. More precisely, we present families of multi-soliton solutions with trivial and nontrivial scattering.

II. Construction of Solitons

The integrable nature of (1) means that there is a variety of methods for constructing solutions. Here, we indicate a general method for constructing soliton solutions of (1) which is a variant of that in Ref. [8]. Using the standard method of Riemann problem with zeros in order to construct the multi-soliton solution, we assume that the function Ψ has the simple form in λ, ie

$$\Psi = I + \sum_{k=1}^{n} \frac{M_k}{\lambda - \mu_k} \tag{13}$$

where M_k are 2×2 matrices independent of λ and n is the soliton number. The components of the matrix M_k are given in terms of a rational function $f_k(\omega_k) = a_k\,\omega_k + c_k$ of the complex variable: $\omega_k = v - r^2\,(\mu_k - u)^{-1}$. Here a_k, c_k and μ_k are complex constants which determine the size, position and velocity of the k-th solitons. *Remark*: The rational dependence of the solutions Ψ follows (directly) when the inverse spectral theory is considered. In [11] (for the flat spacetime), it was shown by solving the Cauchy problem that the spectral data is a function of a parameter similar to (8).

The matrix M_k has the form

$$M_k = \sum_{l=1}^{n} (\Gamma^{-1})^{kl} \bar{m}_a^l m_b^k \tag{14}$$

with Γ^{-1} the inverse of

$$\Gamma^{kl} = \sum_{a=1}^{2} (\bar{\mu}_k - \mu_l)^{-1} \bar{m}_a^k m_a^l \tag{15}$$

and m_a^k holomorphic functions of ω_k, of the form $m_a^k = (m_1^k, m_2^k) = (1, f_k)$. The Yang-Mills-Higgs fields (Φ, A_r, A_v, A_u) can then be read off from (11-12) and they automatically satisfy (10). The corresponding solitons are spatially localized since $\Phi \to 0$ at spatial infinity (ie at $r=0$).

By way of example, let us look at the special case where $\mu_1 = i$, $\mu_2 = 2i$, $a_1 = 2$, $a_2 = 1$, $c_1 = 5$ and $c_2 = -10$. FIG. 2 represents a snapshot of the positive definite gauge quantity $(-\mathrm{tr}\Phi^2)$ at time $t = 8$. The corresponding solution consists of two solitons which travel towards $r = 0$ and bounce back while their sizes change as they move.

III. Scattering Solutions